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Generalized Control Problems with Optimality Conditions under Generalized Univexity

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Abstract:

In this paper, we introduce generalized multi objective programming problems. Further, we will derive optimality condition of Kuhn – Tucker and Fritz – John type under generalized univexity conditions.

Key words: Generalized univexity, Generalized multi objective programming, optimality conditions.

1. Introduction

Convexity and generalized plays an important role in the field of non-linear mathematical programming, game theory, control theory and so on. In their works, Mond et al., [16] established that the invexity of a given functioned is necessary as well as sufficient in its critical points, which were to be global in general as in [16]. Further, Aren et al. [2] introduced a class of functional, known as KT- invex and established that Kuhn-Tne points, which were to be optimal solutions for the chosen control problems. Consequently, Arana et al. [3] generalized these results by weakening the conditions on the involved function known as FJ- Invexity. In another development. Giorgi [10] discussed many different results and gave many remains towards necessary optimality condition of FJ-type of a non linear programming with inequality and equality constrains again, Hussain et al., [13] established sufficient FJoptimality conditions when the objective function was pseudo-convex and the corresponding constraint functions were quasi convex. In another work, Flores-Bazan [9] discussed an alternative-type of Fritz-John optimality conditions. Further, Slimani et.al [26] studied a generalized FJ-Condition, which was both necessary as well as sufficient for a feasible point under generalized invexity conditions. Subsequently Singh et.al [25] considered order relationship between the closed interval in real numbers and established both theoretical and practical solution approaches for multi objective programming problems by making use of interval valued objective function.

Very recently, Pandey and Mishra [18] introduced a new concept called in – stationary point for considered non-smooth multi objective semi-infinite mathematical programming problem with equilibrium constraints by making use of clarice sub differentials and obtained KKT type optimality condition. In the sequel, Pitea et al [22, 23, 24] developed some applications to applied sciences by introducing a class of multi objective optimization problem under generalized invexity condition [1, 2, 3, 5, 6, 8, 15, 25]. Also, in a very recent development Padhon et al. [17] introduced control problems with generalized invexity and obtained optimality condition of the type Kuhn-Tucker at Fritz John.

By making use of the above ideas, here we generalize the results of Padhan et al [17] by weakening the convexity conditions involved on the objective as well as constraint function.

Hence, we introduce KT-ρ-(ξ,η,θ) invexity and FJ-ρ-(ξ,η,θ)-invexity because such functions have many different applications in control design for autonomous vehicles, [12] optimal control of static elastoplasticity [14] electrical power production. [24], economy [14] medicine [27] ecology [23] computer integrated mann factory Robotics [17] and Wavelet analysis [12]. In section 2, we introduced generalized multi objective mathematical control problem and preliminary notation and definition. In section 3, we will study optimality conditions.

2. Notations and Definitions

Here, we consider a multi objective control problem (MCP) as follows.

$$\text{(MCP)} \qquad H(x, u) = \int_b^a f_i\left(t, x, \dot{x}, u, \dot{u}\right) dt$$

subject to condition

$$x(a) = \alpha, \ x(b) = \beta$$

$$g_j\left(t, x, \dot{x}, u, \dot{u}\right) \le 0 \ h_k\left(t, x, \dot{x}, u, \dot{u}\right) = x, \quad t \in I = [a, b].$$

Here the interval $I = [a, b]$ is a real one, $f : z \times p^n \times R^n \to R$, $g_j : I \times R^n \times R^m \to R^K$ and $h_k : I \times R^n \times R^m \to R^n$ are continuously differentiable functions. The corresponding partial derivatives of f w.r.t. x and t are

$$f_t : \frac{\partial f}{\partial t}, f_x = \left(\frac{\partial f_1}{\partial x_1}, \frac{\partial f_2}{\partial x_2} .... \frac{\partial f_n}{\partial x_n}\right)$$

$$f_u = \left(\frac{\partial f_1}{\partial u_1}, \frac{\partial f_2}{\partial u_2}, ..., \frac{\partial f_n}{\partial u_m}\right)$$

Similarly, the corresponding partial derivatives of constraint function g_j and h_k can be defined and denoted by g_t, g_u, g_k at h_t, h_x, h_u and X is represent the state of piece wise smooth state functions $x : I = (a, b) \to R^n$ such that x (a) = α and $x(b) = \beta$ which is equipped with required norm $\|x\| = \|x\|_\alpha + \|D_x\|_\infty$. Consequently, γ is the space of piecewise continuous control functions $u : I = [a, b] \to R^m$, Further, the corresponds feasible solution of the multi objective control problem (MCP) lies in the university set, which is defined as:

$$K_1 = \left\{x \in X : y \in Y : x(a) = \alpha, \ x(b) = \beta, g_j\left(t, x, \dot{x}, u, \dot{u}\right) \le o, h_k\left(t, x, \dot{x}, u, \dot{u}\right), I \in [a, b]\right\}$$

The following definition is needed in the sequel in our work.

Definition 2.1:

A point $(\tilde{x},\tilde{u}) \in K_1$ is know as Kuhn-Tucker point if $\exists$ piecewise smooth functions $\lambda_i : I=[a,b] \to R^K$ and $\mu_j : I[a,b] \to R^n$ satisfying the following condition.

$$\frac{\partial f_i}{\partial x}(t,x,\dot{x},u,\dot{u}) + \lambda_i(t)^T \frac{\partial g_j}{\partial x}(t,x,\dot{x},u,\dot{u})$$

$$+\mu_j(t)^T \frac{\partial h_x}{\partial x}(t,x,\dot{x},u,\dot{u}) + \mu_j(t) = 0$$

$$\frac{\partial f_i}{\partial u}(t,x,\dot{x},u,\dot{u}) + \lambda_i(t)^T \frac{\partial g_j}{\partial u}(t,x,\dot{x},u,\dot{u}) = 0$$

$$\lambda_j(t)^T g_j(t,x,\dot{x},u,\dot{u}) = 0$$

$\lambda_j(t) \geq 0$, for all $t \in I = (a,b)$, except at the discontinuities.

The following definition, which is useful in the sequal from Arana et al. [2, 3]

Definition 2.2. The problem (MCP) is known as KT-ρ-(ξ-η-θ) univex at the point $(x_0,u_0) \in K$, if for all $(x_{0,}u_0)$ $\in$K, and for all $\lambda : I=[a,b] \to e^K$, which satisfier $\lambda_1(t)^T g_j(t,x_0,u_0)=0$, $\lambda_1(t) \geq 0$ and $\mu_1 : I=[a,b] \to R^n$ piecewise smooth function, $\exists$ differentiable vector point function

$\eta^T(t,x,x_0,u,u_0,\lambda_1,\mu_1)$ and $\lambda_1{}^T(t,x,x_0,u,u_0,\lambda_1,\mu_1)$

and a corresponding vector point function

$\theta_1(t,x,x_0,u,u_0,\lambda_1,\mu_1)$ with respective

$\rho_1 \| \theta_1(t,x,x_0,u,u_0,\lambda_1,\mu_1) \|^2$

$\geq 0, \rho_1 \in R$ such that,

$$b(x,u)H\{(x,u)-H(x_0,u_0)\}\Rightarrow$$

$$\int_a^b\left[\left\{\frac{\partial f_i}{\partial x}(t,x_0,\dot{x}_0,u_0,\dot{u}_0)+\lambda_1^T(t)\frac{\partial g_j}{\partial x}(t,x_0,\dot{x}_0,u_0,\dot{u}_0)\right\}\right]$$

$$\int_a^b\left[\left\{\frac{\partial f_i}{\partial x}(t,x_0,\dot{x}_0,u_0,\dot{u}_0)+\lambda_1^T(t)\frac{\partial g_j}{\partial x}(t,x_0,\dot{x}_0,u_0,\dot{u}_0)+\mu_1^T\frac{\partial h_k}{\partial x}(t,x_0,\dot{x}_0,u_0,\dot{u}_0)\right\}\right]$$

$$-\mu_1^T(t)\eta^T(t,x_0,\dot{x}_0,u_0,\dot{u}_0)$$

$$+\left\{\frac{\partial f_i}{\partial u}(t,x_0,\dot{x}_0,u_0,\dot{u}_0)+\lambda_1^T\frac{\partial g_j}{\partial u}(t,x_0,\dot{x}_0,u_0,\dot{u}_0)+\mu_1^T(t)\frac{\partial h_k}{\partial u}(t,x_0,\dot{x}_0,u_0,\dot{u}_0)\right\}$$

$$+\rho_1 \lVert\theta_1(t,x_0,x,y_0,y,\lambda_1,\mu_1)\rVert^2 < 0.$$

The problem (MCP) is known to be KT-ρ_1-(ξ_1, η_1,μ_1) invex it if is true for all $(x_0,y_0)\in K_1$.

3. Results

The following optimality conditions are generalized from of different of results available in the literature.

Theorem 3.1:

If (MCP) is KT-ρ_1-(ξ_1, η_1,μ_1) univex, then all Kuhn-Tucker point, and optimal solutions for (MCP).

Proof: Suppose that (MCP) is KT-ρ_1-(ξ_1, η_1,μ_1) -invex.

Let (x_0,y_0) be a Kuhn-Tucker point, then $\exists\partial_1 : I=[a,b]\rightarrow R^K$ and $\mu_1 : I=(a,b)\rightarrow R^n$ satisfying the optimality conditions (4) to (7).

Consider

$$\int_a^b \left[\left\{ \frac{\partial f_i}{\partial x}(t, x_0, \dot{x}_0, u_0, \dot{u}_0) + \lambda_1^T(t) \frac{\partial g_j}{\partial x}(t, x_0, \dot{x}_0, u_0, \dot{u}_0) + \mu_1^T(t) \frac{\partial h_k}{\partial x}(t, x_0, \dot{x}_0, u_0, \dot{u}_0) \right\} \right]$$

$$-\mu_1^T(t)\eta^T(t, x, x_0, y, y_0, \lambda_1, \mu_1)$$

$$+\left\{ \frac{\partial f_i}{\partial x}(t, x, x_0, y, y_0, \lambda_1, \mu_1) + \lambda_1^T(t) \frac{\partial g_j}{\partial x}(t, x, x_0, y, y_0, \lambda_1, \mu_1) \right\}$$

$$+\mu_1^T(t) \frac{\partial h_k}{\partial x}(t, x, x_0, y, y_0, \lambda_1, \mu_1)$$

$$+\rho_1 \left\| \theta_1(t, x, x_0, y, y_0, \lambda_1, \mu_1) \right\|^2$$

Since the problem (MCP) is $KT - \rho_1 - (\xi_1, \eta_1, \theta_1)$ univexity, by definition, we obtain

$$b \; b(x,u)\left(H(x,u) - H(\tilde{x}, \tilde{u})\right) \geq 0, \; \forall \; (x,u) \in K_1.$$

Hence, $(\tilde{x}, \tilde{u})$ is also an optimal solution for the problem (MCP)

Hence proved. □

The above conditions are not only sufficient but necessary also in some cases.

Theorem 3.2. All the KT-Points are optimal solutions for the problems (MCP), then the problem (MCP) is $KT - \rho_1 - (\xi_1 - \eta_1 - \theta_1) - \text{invex}$.

Proof:

Suppose $(x_o, y_o) \in K_1$ is a KT-Point, $\exists$ a piecewise smooth functions $\lambda_1 = [a,b] \to R^K$ and $\mu_1 : I = [a,b] \to R^n \ni$ the KT-necessary are satisfied.

Since, $\forall (x_o, u_o) \in K_1$ and the KT-points are optimal then by definition, we have

$$b(x_o, y_o)\,(H(x_o, y_o) - H\,(\tilde{x}, \tilde{y})) \geq 0$$

This result can be proved by using contradiction.

Let us assume that eh problem (MCP) is not

$KT - \rho_1 - (\xi_1 - \eta_1 - \theta_1) - \text{invex}.$

Then

$$\int_a^b \left[\left\{ \frac{\partial f_i}{\partial x}(t,\ x,\ \tilde{x},\ u,\ \tilde{u}) + \lambda(t)^T \frac{\partial g_j}{\partial x}(t,\ x,\ \tilde{x},\ u,\ \tilde{u}) \right. \right.$$

$$+\mu(t)^T \frac{\partial h_k}{\partial u}(t,\ x,\ \tilde{x},\ u,\ \tilde{u})\ \eta^T(t,\ x,\ \tilde{x},\ u,\ \tilde{u},\ \lambda_1, \mu_1) - \mu(t)^T\ \eta^T(t,\ x,\ \tilde{x},\ u,\ \tilde{u},\ \lambda_1, \mu_1)$$

$$+\left\{ \frac{\partial f_i}{\partial u}(t,\ x,\ \tilde{x},\ u,\ \tilde{u}) + \lambda(t)^T \frac{\partial g_j}{\partial u}(t,\ x,\ \tilde{x},\ u,\ \tilde{u}) \right\}$$

$$\left. \left. +\mu(t)^T \frac{\partial h_k}{\partial u}(t,\ x,\ \tilde{x},\ u,\ \tilde{u}) \right\} \xi_1(t,\ x,\ \tilde{x},\ u,\ \tilde{u}, \lambda_1, \mu_1) \right]$$

$$+\rho_1 \| \theta(t,\ x,\ \tilde{x},\ u,\ \tilde{u}, \lambda_1, \mu_1) \|^2 \geq 0$$

$$\Rightarrow b(x_o, y_o) H((x_o, y_o) - H(\tilde{x}, \tilde{y})) > 0$$

which is a contradiction to our hypothesis.

Hence proved. □

Conclusions

In this paper we derived generalized KKT- optimality conditions and duality theorems with respective to generalized KT-ρ-(ξ,η,θ) invexity. These results are generalizations of S.K. Padhan et al., [17].

References

[1]. Antczak T. and Pitea A., "Parametric approach to multitime multiobjective fractional variational problems under (F, ρ)-convexity", Optim. Control Appl. Methods (2015).

[2]. Arana M., Osuna R., Rufian A. and Ruiz G., KT-invex control problem. Appl. Math. Comput. 197 (2008) 489-496.

[3]. Arana M., Osuna R., Rufian A. and Ruiz G., FJ-invex control problem. Appl. Math. Lett. 22 (2009) 1887-1891.

[4]. Betz T. and Meyer C., Second-order sufficient optimality conditions for optimal control of static elastoplasticity with hardening. ESAIM Control Optim. Calc. Var. 21 (2015) 271-300.

[5]. Chandra S., Craven B.D. and Husain I., A class of non-differentiable control problems. J. Optim. Theory Appl. 56 (1988) 227-243.

[6]. Christensen G.S., El-Hawary M.E. and Soliman S.A., Optimal control applications in electric power system. New York, Plenum Press, 1987.

[7]. Craven B.D., Mathematical Programming and Control Theory. Chapman and Hall, London, 1978.

[8]. Dakev N.V., Chipperfield A.J. and Fleming P.J., Intelligent Systems for the 21st Century. Proc. IEEE Int. Conf. Syst. Man Cyber. 5 (1995) 4503-4508.

[9]. Flores-Baz´an F., Fritz John necessary optimality conditions of the alternative-type. J. Optim. Theory Appl. 161 (2014) 807-818.

[10]. Giorgi G., Remarks on Fritz John conditions for problems with inequality and equality constraints. Int. J. Pure Appl. Math. 71 (2011) 643-657.

[11]. Hanson M.A., On sufficiency of Kuhn-Tucker conditions. J. Math. Anal. Appl. 80 (1981) 545-550.

[12]. Heydari M.H., Hooshmandasl M.R., Maalek Ghaini F.M. and Cattani C., Wavelets method for solving fractional optimal control problems. Appl. Math. Comput. 286 (2016) 139-154.

[13]. Husain I. and Srivastav S.K., Sufficient Fritz John type optimality criteria and duality for control problems. Amer. J. Ind. Business Man. 3 (2013) 237-244.

[14]. Intriligator M.D., Mathematical Optimization and Economic Theory. Prentice-Hall, Englewood Cliffs, New Jersey, 1971.

[15]. Leitmann G., The Calculus of Variations and Optimal Control. New York, Plenum Press, 1981.

[16]. Mond B. and Smart I., Duality and sufficiency in control problems with invexity. J. Math. Anal. Appl. 136 (1988) 325-333.

[17]. Padhan S.K., Chandal Nahak, Mihai Postolache, “Control Problems With Kuhn-Tucker And Fritz John Generalized Invexity”, Journal of Mathematical Analysis, Volume 7 Issue 3, pp: 51-57, 2016.

[18]. Pandey Y. and Mishra S.K., On strong KKT type sufficient optimality conditions for nonsmooth multiobjective semi-infinite mathematical programming problems with equilibrium constraints. Oper. Res. Lett. 44 (2016) 148-151.

[19]. Pereira F.L., A maximum principle for impulsive control problems with state constraints. Int. J. Comput. Appl. Math. 19 (2000) 1-19.

[20]. Pereira F.L., Control design for autonomous vehicles: A dynamic optimization perspective. Eur. J. Control 7 (2001) 178-202.

[21]. Pitea A. and Antczak T., Proper efficiency and duality for a new class of nonconvex multitime multiobjective variational problems, J. Inequal. Appl. 2014, Art. No. 333.

[22]. Pitea A., Postolache M., Duality theorems for a new class of multitime multiobjective variational problems. J. Glob. Optim. 54 (2012) 47-58.

[23]. Pitea, A, Postolache, M: Minimization of vectors of curvilinear functionals on the second order jet bundle. Necessary conditions. Optim. Lett. 6(2012), No. 3, 459-470.

[24]. Pitea, A, Postolache, M: Minimization of vectors of curvilinear functionals on the second order jet bundle. Sufficient efficiency conditions. Optim. Lett. 6(2012), No. 8, 1657-1669.

[25]. Singh D., Dar B.A. and Kim D.S., KKT optimality conditions in interval valued multiobjective programming with generalized differentiable functions. Eur. J. Oper. Res. 254 (2016) 29-39.

[26]. Slimani H. and Radjef M.S., Generalized Fritz John optimality in nonlinear programming in the presence of equality and inequality constraints. Oper. Res. Int. J. (2015).

[27]. Swam G.W., Applications of Optimal Control Theory in Biomedicine. Marcel Dekker, New York, 1984.